§3. Rings and Modules of fractions

$\mathbb{Z} \mapsto \mathbb{Q} = \left\{ \frac{a}{s} \,\middle|\, a, s \in \mathbb{Z}, s \neq 0 \right\}$

$$\frac{a}{s} = \frac{b}{t} \Leftrightarrow at = bs$$

field of fractions Let A be an integral domain.

Consider "$\equiv$" on $A \times A \setminus \{0\}$ defined by

$$(a, s) \equiv (b, t) \Leftrightarrow at - bs = 0$$

$$\mathrm{Frac}\, A = (A \times A \setminus \{0\}) /_{\equiv}$$

$$= \left\{ \frac{a}{s} \,\middle|\, \begin{array}{l} a \in A,\ s \in A \setminus \{0\} \\ \frac{a}{s} := \text{the equ. class of } (a, s) \end{array} \right\}$$

更一般的有:

§3.1 ring of fractions of A with respect to S

A = ring

A <u>multiplicatively closed subset</u> of A is a subset S of A s.t.

1) $1 \in S$
2) $a, b \in S \Rightarrow ab \in S$.

Let S be a mult. closed subset of A.

"$\equiv$" on $A \times S$:

$$(a,s) \equiv (b,t) \overset{\text{def}}{\Longleftrightarrow} \exists u \in S \text{ s.t. } (at-bs)u = 0$$

- reflex ✓
- symmetric ✓
- transitive ?

Pf: $(a,s) \equiv (b,t) \Rightarrow (at-bs)v = 0$

$(b,t) \equiv (c,u) \Rightarrow (bu-ct)w = 0$

$$\xRightarrow{\text{eliminate } b} (au-cs)tvw = 0$$

$$S^{-1}A := (A \times S) / \equiv$$

$$= \left\{ \frac{a}{s} \;\middle|\; a \in A,\ s \in S \right\}$$

equivalence class of (a,s).

"+" $\quad \frac{a}{s} + \frac{b}{t} = \frac{at+bs}{st}$

"$\cdot$" $\quad \frac{a}{s} \cdot \frac{b}{t} = \frac{ab}{st}$

Fact: i) $(S^{-1}A, +, \cdot)$ = ring with identity $\frac{1}{1}$.

ii) ring homo. $f : A \to S^{-1}A \quad a \mapsto \frac{a}{1}$

(not inj. in general)

iii) $S^{-1}A = \mathrm{Frac}\,A$, if A = int. domain and $S = A \setminus \{0\}$.

Prop 3.1 (Universal property of $S^{-1}A$)

$S \subset A$ mult. closed subset

If $g: A \to B$ ring hom. with $g(S) \subseteq B^\times$,

then $\exists!\ h: S^{-1}A \to B$ s.t. $g = h \circ f$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g"'] & S^{-1}A \arrow[d, dashed, "\exists!\, h"] \\ & B \end{tikzcd}$$

Pf: Uniqueness:

$$h\left(\frac{a}{s}\right) = h\left(\frac{a}{1}\right) h\left(\frac{s}{1}\right)^{-1} = g(a) \cdot g(s)^{-1}$$

existence:

$$h\left(\frac{a}{s}\right) := g(a)\, g(s)^{-1}$$

ONTS: h is well-defined.

$$\frac{a}{s} = \frac{a'}{s'} \Rightarrow (as' - a's)t = 0$$

$$\Rightarrow \left(g(a)\, g(s') - g(a')\, g(s)\right) g(t) = 0$$

$$\overset{g(t) \in B^\times}{\Rightarrow} g(a)\, g(s)^{-1} = g(a')\, g(s')^{-1}$$

$\square$

Facts: $f: A \to S^{-1}A$

i) $s \in S \Rightarrow f(s) \in (S^{-1}A)^{\times}$

ii) $f(a) = 0 \Rightarrow \exists\, s \in S$ s.t. $sa = 0$

iii) $S^{-1}A = \{ f(a) \cdot f(s)^{-1} \mid a \in A, s \in S \}$

Cor 3.2 (characterize $S^{-1}A$) $g: A \to B$ ring hom. If

i) $s \in S \Rightarrow g(s) \in B^{\times}$

ii) $g(a) = 0 \Rightarrow \exists\, s \in S$ s.t. $sa = 0$

iii) $B = \{ g(a)\, g(s)^{-1} \mid a \in A, s \in S \}$

then $\exists!$ iso. $h : S^{-1}A \xrightarrow{\sim} B$ s.t. $g = h \circ f$.

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "g"'] & S^{-1}A \arrow[d, "h"] \\
& B
\end{tikzcd}
$$

Pf: WTS: h = iso.

- iii) $\Rightarrow h$ = surj
- $\forall\ \frac{a}{s} \in \ker h \Rightarrow a \in \ker g$

$\Rightarrow \exists\, t \in S$ s.t. $at = 0$

$\Rightarrow \frac{a}{s} = \frac{0}{1} = 0 \in S^{-1}A$.

Example:

1) localization of A at $\mathfrak{p}$:

$\mathfrak{p}$ = prime ideal of A

$S := A - \mathfrak{p}$

$A_{\mathfrak{p}} := S^{-1}A$ is a local ring with maximal ideal $S^{-1}\mathfrak{p} := \{ \frac{a}{s} \in A_{\mathfrak{p}} \mid a \in \mathfrak{p},\ s \in S \}$.

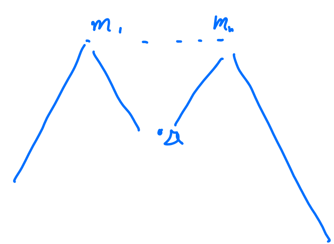
2) $0 \in S \Leftrightarrow S^{-1}A = 0$

3) $A_f := S^{-1}A$ where $S = \{1, f, f^2, \cdots\}$

4) $S^{-1}\mathfrak{b} \neq S^{-1}A$

$\Leftrightarrow \mathfrak{a} + \mathfrak{b} \neq (1)$

$\Leftrightarrow \mathfrak{b} \subseteq m \supseteq \mathfrak{a}$ for some m



$$\mathrm{Rad}(S^{-1}A) \supseteq S^{-1}\mathfrak{a}$$

5.i) $A = \mathbb{Z}$, $\mathfrak{p} = (p)$

$A_{(p)} = \{ \frac{m}{n} \in \mathbb{Q} \mid (n, p) = 1 \}$

$A_p = \{ \frac{m}{p^n} \in \mathbb{Q} \}$

ii) $A = k[t_1, \dots, t_n]$ $\quad \mathfrak{p} \triangleleft A$ prime

$V = V(\mathfrak{p}) \subseteq \mathbb{A}^n = k^n \qquad g \in \mathfrak{p} \Leftrightarrow g|_V = 0.$

$$A_{\mathfrak{p}} = \left\{ \frac{f}{g} \in k(t_1 \dots t_n) \;\middle|\; g \notin \mathfrak{p} \right\}$$

$$= \left\{ \frac{f}{g} \;\middle|\; g|_V \neq 0 \right\}$$

local ring of k^n along V

- Let M be an A-module.

$$S^{-1}M := M \times S / \equiv$$

- $(m, s) \equiv (m', s') \overset{\text{def}}{\Longleftrightarrow} \exists t \in S$ s.t. $t(sm' - s'm) = 0$
- $S^{-1}A$-module str.: $\dfrac{a}{s'} \cdot \dfrac{m}{s} := \dfrac{am}{s's}$ (well-defined)

- $M_{\mathfrak{p}} := S^{-1}M \quad S = A - \mathfrak{p}$
- $M_f := S^{-1}M \quad S = \{1, f, f^2, \cdots\}$

- Let $u : M \to N$ be an A-module hom.

$$S^{-1}u : S^{-1}M \longrightarrow S^{-1}N \qquad S^{-1}A\text{-mod. hom.}$$

$$\frac{m}{s} \longmapsto \frac{u(m)}{s}$$

Fact: $S^{-1}(v \circ u) = (S^{-1}v) \circ (S^{-1}u)$

Prop 3.3. S^{-1} is exact.

i.e. $M' \xrightarrow{f} M \xrightarrow{g} M''$ exact $\Rightarrow S^{-1}M' \xrightarrow{S^{-1}f} S^{-1}M \xrightarrow{S^{-1}g} S^{-1}M''$ exact pf: 1° $g\circ f = 0 \Rightarrow S^{-1}g \circ S^{-1}f = S^{-1}(0) = 0$

$\Rightarrow \operatorname{Im}(S^{-1}f) \subseteq \ker S^{-1}g$

2° $\forall\ \frac{m}{s} \in \ker S^{-1}g$

$\Rightarrow \frac{g(m)}{s} = \frac{0}{1} \in S^{-1}M''$

$\Rightarrow \exists\, t \in S$ s.t. $g(tm) = t\,g(m) = 0$

$\Rightarrow tm \in \ker g = \operatorname{im} f$

$\Rightarrow tm = f(m')$ for some m'

$\Rightarrow \frac{m}{s} = \frac{tm}{ts} = \frac{f(m')}{ts} = S^{-1}f\left(\frac{m'}{ts}\right)$

$\in \operatorname{im}(S^{-1}f)$

$\Rightarrow \ker S^{-1}g \subseteq \operatorname{im} S^{-1}f$ $\square$

$M' \subseteq M \Rightarrow S^{-1}M' \hookrightarrow S^{-1}M$ regarded as a submodule

Cor 3.4 S^{-1} commutes with $+$, $\cap$, quotients.

i.e. Let $N, P \subseteq M$ be submodules. Then

i) $S^{-1}(N+P) = S^{-1}(N) + S^{-1}(P)$ $\quad (\subset S^{-1}M)$

ii) $S^{-1}(N \cap P) = S^{-1}(N) \cap S^{-1}(P)$ $\quad (\subset S^{-1}M)$

iii) $S^{-1}(M/N) \cong S^{-1}(M)/S^{-1}(N)$

Pf: i) definition

ii) "$\subseteq$" clear.

"$\supseteq$". $\forall$ $\frac{y}{s} = \frac{z}{t}$ $\Rightarrow u(yt - sz) = 0$ $\Rightarrow w = uyt = usz \in N \cap P$

$\Rightarrow \frac{y}{s} = \frac{w}{stu} \in S^{-1}(N \cap P)$.

iii) $0 \to N \to M \to M/N \to 0$ $\Rightarrow$ ✓

Prop 3.5 $S^{-1}A \otimes_A M \xrightarrow{\sim} S^{-1}M$

$\frac{a}{s} \otimes m \longmapsto \frac{am}{s}$

Pf: Consider $S^{-1}A \times M \longrightarrow S^{-1}M$

$$\left(\frac{a}{s}, m\right) \longmapsto \frac{am}{s}$$

- A-bilinear

$$\Rightarrow \exists! \, f : S^{-1}A \otimes_A M \longrightarrow S^{-1}M$$

$$\frac{a}{s} \otimes m \longmapsto \frac{am}{s}$$

- surj. (clear)

- inj.

$$\forall \; x = \sum_i \frac{a_i}{s_i} \otimes m_i \in S^{-1}A \otimes M$$

$$s := \prod_i s_i, \quad t_i := \prod_{j \neq i} s_j \quad \left(s_i = \frac{s}{t_i}\right)$$

$$\Rightarrow \; x = \sum_i \frac{a_i t_i}{s} \otimes m_i = \frac{1}{s} \otimes \sum_i a_i t_i m_i$$

$\Rightarrow$ any element in $S^{-1}A \otimes M$ is of form $\frac{1}{s} \otimes m$

$\forall\ \frac{1}{s}\otimes m \in \ker f \Rightarrow \frac{m}{s} = 0$

$\Rightarrow \exists\, t \in S$ s.t. $tm = 0$

$\Rightarrow \frac{1}{s}\otimes m = \frac{t}{st}\otimes m = \frac{1}{st}\otimes tm = 0$

<u>Cor</u> 3.6. $S^{-1}A$ is a flat A-module.

<u>Prop</u> 3.7 $S^{-1}M \otimes_{S^{-1}A} S^{-1}N \xrightarrow{\sim} S^{-1}(M \otimes_A N)$

Pf: $S^{-1}M \otimes_{S^{-1}A} S^{-1}N \cong (S^{-1}A \otimes_A M) \otimes_{S^{-1}A} (S^{-1}A \otimes_A N)$

$\cong M \otimes_A (S^{-1}A \otimes_{S^{-1}A} S^{-1}A) \otimes_A N$

$\cong S^{-1}A \otimes (M \otimes_A N)$

$\cong S^{-1}(M \otimes_A N)$

<u>Cor</u>: $M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} N_{\mathfrak{p}} = (M \otimes_A N)_{\mathfrak{p}}$

§ 3.2 local properties

Def: A property P of a ring A (or of an A-module) is said to be a local property if

A (or M) has P $\Leftrightarrow$ $A_{\mathfrak{p}}$ (or $M_{\mathfrak{p}}$) has P $\forall \mathfrak{p}$

Prop 3.8. $M = A$-module. TFAE

i) $M = 0$

ii) $M_{\mathfrak{p}} = 0$ $\forall \mathfrak{p}$ prime ideals of A

iii) $M_{\mathfrak{m}} = 0$ $\forall \mathfrak{m}$ maximal ideals of A

Pf: i) $\Rightarrow$ ii) $\Rightarrow$ iii) clear

iii) $\Rightarrow$ i) : Suppose $M \neq 0$. $\forall x \in M \setminus \{0\}$.

$$\mathfrak{a} := \mathrm{Ann}(x) \neq A$$

$$\Rightarrow \exists\, \mathfrak{m} \text{ s.t. } \mathfrak{a} \subseteq \mathfrak{m}$$

$M_m = 0 \Rightarrow \frac{x}{1} = 0 \Rightarrow \exists\, a \in A - m$ s.t.

$ax = 0$

$\Rightarrow \mathfrak{a} \cap A - m \neq \phi$ ↯

<u>Prop</u> 3.9. $\phi : M \to N$ A-mod. hom. TFAE

i) $\phi =$ inj

ii) $\phi_{\mathfrak{p}} =$ inj $\forall \mathfrak{p}$

iii) $\phi_m =$ inj $\forall m$

Similar for surjective.

Pf: $\ker(\phi_{\mathfrak{p}}) = (\ker \phi)_{\mathfrak{p}}$

$$\begin{array}{ccccc} \ker\phi = 0 & \Leftrightarrow & \ker\phi_{\mathfrak{p}} = 0 \ \forall \mathfrak{p} & \Leftrightarrow & \ker\phi_m = 0 \ \forall m \\ \Updownarrow & & \Updownarrow & & \Updownarrow \\ \phi = \text{inj} & & \phi_{\mathfrak{p}} = \text{inj} \ \forall \mathfrak{p} & & \phi_m = \text{inj} \ \forall m \end{array}$$

Prop 3.10 Flatness is a local property. i.e. TFAE

i) M/A = flat

ii) $M_{\mathfrak{p}}/A_{\mathfrak{p}}$ = flat $\forall \mathfrak{p}$

iii) $M_{\mathfrak{m}}/A_{\mathfrak{m}}$ = flat $\forall \mathfrak{m}$.

pf: i) $\Rightarrow$ ii) $M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} - = (M \otimes_A A_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} -) = M \otimes_A -$

ii) $\Rightarrow$ iii) clear

iii) $\Rightarrow$ i): $\forall N \hookrightarrow P$

$$\Rightarrow N_{\mathfrak{m}} \hookrightarrow P_{\mathfrak{m}} \quad \forall \mathfrak{m}$$

$$\Rightarrow M_{\mathfrak{m}} \otimes_{A_{\mathfrak{m}}} N_{\mathfrak{m}} \hookrightarrow M_{\mathfrak{m}} \otimes_{A_{\mathfrak{m}}} P_{\mathfrak{m}} \quad \forall \mathfrak{m}$$

$$\Rightarrow (M \otimes_A N)_{\mathfrak{m}} \hookrightarrow (M \otimes_A P)_{\mathfrak{m}} \quad \forall \mathfrak{m}$$

$$\Rightarrow M \otimes_A N \hookrightarrow M \otimes_A P$$

$\Rightarrow M \otimes -$ exact $\Rightarrow M$ = flat.

§3.3 extended and contracted ideals in rings of fractions

$f: A \to S^{-1}A \quad a \mapsto \frac{a}{1}$

$C := \{ \text{contracted ideals in } A \}$

$\updownarrow$ 1:1

$E := \{ \text{extended ideals in } S^{-1}A \}$

Fact : $\mathfrak{a}^e = S^{-1}\mathfrak{a}$

$s = s_i t_i$

pf: $\sum_i f(a_i) \cdot \frac{b_i}{s_i} = \sum_i \frac{a_i b_i}{s_i} = \frac{\sum_i a_i b_i t_i}{s} =: \frac{a}{s}$

Prop 3.11 i) $\mathfrak{b} \lhd S^{-1}A \Rightarrow \mathfrak{b} \in E$

i.e. $\forall \mathfrak{b} \lhd S^{-1}A \ \exists \mathfrak{a} \lhd A$ s.t. $\mathfrak{b} = S^{-1}\mathfrak{a}$

ii) $\mathfrak{a} \lhd A \Rightarrow \mathfrak{a}^{ec} = \bigcup_{s \in S} (\mathfrak{a} : s)$

in particular : $\mathfrak{a}^e = (1) \Leftrightarrow \mathfrak{a} \cap S \neq \phi$

iii) $\mathfrak{a} \triangleleft A$.

$\mathfrak{a} \in C \Leftrightarrow \forall s \in S$ is not zero-divisor in $A/\mathfrak{a}$

iv) $\{\mathfrak{p} \triangleleft A : \text{prime} \mid \mathfrak{p} \cap S = \phi\} \xleftrightarrow{1:1} \{\mathfrak{q} \triangleleft S^{-1}A : \text{prime}\}$

v) S^{-1} commutes with Σ, Π, $\cap$ & $\sqrt{\ }$

pf: i) $\mathfrak{b} \triangleleft S^{-1}A$,

$\forall \frac{x}{s} \in \mathfrak{b}$

$\Rightarrow \frac{x}{1} = \frac{s}{1} \cdot \frac{x}{s} \in \mathfrak{b}$

$\Rightarrow x \in \mathfrak{b}^c$

$\Rightarrow \frac{x}{s} \in \mathfrak{b}^{ce}$

$\Rightarrow \mathfrak{b} \subseteq \mathfrak{b}^{ce} \Rightarrow \mathfrak{b} = \mathfrak{b}^{ce} = (\mathfrak{b}^c)^e$

ii) $x \in \mathfrak{a}^{ec} = (S^{-1}\mathfrak{a})^c$

$\Leftrightarrow \frac{x}{1} \in S^{-1}\mathfrak{a}$

$$\Leftrightarrow \quad \frac{x}{1} = \frac{a}{s'} \quad \text{for some } a \in \mathfrak{a},\ s' \in S$$

$$\Leftrightarrow \quad (xs' - a)t = 0 \quad \text{for some } a \in \mathfrak{a},\ s', t \in S$$

$$\xrightarrow{s = s't}$$

$$\Leftrightarrow \quad x \cdot s \in \mathfrak{a} \quad \text{for some } s \in S$$

$$\xleftarrow[a = xs,\ t = 1]{}$$

$$\Leftrightarrow \quad x \in (\mathfrak{a} : s) \quad \text{for some } s \in S$$

iii) $\mathfrak{a} \in C \Leftrightarrow \mathfrak{a}^{ec} \subseteq \mathfrak{a}$

$$\Leftrightarrow \left(\frac{x}{1} \in \mathfrak{a}^e \Rightarrow x \in \mathfrak{a} \right)$$

$$\Leftrightarrow \left(sx \in \mathfrak{a} \overset{s \in S}{\Longrightarrow} x \in \mathfrak{a} \right)$$

$$\Leftrightarrow \left(\bar{s}\bar{x} = 0 \text{ in } A/\mathfrak{a} \overset{s \in S}{\Longrightarrow} \bar{x} = 0 \text{ in } A/\mathfrak{a} \right)$$

$$\Leftrightarrow \quad \bar{s} \text{ is not zero divisor in } A/\mathfrak{a} \quad \forall s \in S$$

iv)

$$\begin{array}{ccc} C & \overset{1:1}{\longleftrightarrow} & E \\ \cup\ \text{(iii)} & & \cup\ \text{(i)} \\ \{ \mathfrak{p} \triangleleft A \text{ prime} \mid \mathfrak{p} \cap S = \phi \} & \underset{?}{\overset{f^{-1}(\mathfrak{q}) \hookleftarrow \mathfrak{q}}{\rightleftarrows}} & \{ \mathfrak{q} \triangleleft S^{-1}A \text{ prime} \} \end{array}$$

$$\mathfrak{p} \mapsto S^{-1}\mathfrak{p}$$

$$A/f^{-1}(q) \hookrightarrow S^{-1}A/q \Rightarrow f^{-1}(q) = \text{prime in } A$$

$$S^{-1}A / S^{-1}\mathfrak{p} = \bar{S}^{-1}(A/\mathfrak{p}) \subseteq \mathrm{Frac}(A/\mathfrak{p})$$

$$\Rightarrow S^{-1}\mathfrak{p} = \text{prime in } S^{-1}A$$

v) Σ & π ✓ (1.18) $\cap$ ✓ (3.4)

$$S^{-1}(\sqrt{\mathfrak{a}}) \subseteq S^{-1}\Big(\bigcap_{\mathfrak{p} \supseteq \mathfrak{a}} \mathfrak{p}\Big) = \bigcap_{\mathfrak{p} \supset \mathfrak{a}} S^{-1}\mathfrak{p}$$

$$= \bigcap_{q \supset S^{-1}\mathfrak{a}} q = \sqrt{S^{-1}\mathfrak{a}}$$

$$\forall \ \frac{r}{s} \in \sqrt{S^{-1}\mathfrak{a}} \Rightarrow \frac{r^n}{s^n} = \frac{a}{t}$$

$$\Rightarrow u(tr^n - as^n) = 0$$

$$\Rightarrow (utr)^n \in \mathfrak{a}$$

$$\Rightarrow utr \in \sqrt{\mathfrak{a}}$$

$$\Rightarrow \frac{r}{s} = \frac{utr}{sut} \in S^{-1}\sqrt{\mathfrak{a}}$$

$\sqrt{0} = \bigcap_{\mathfrak{p}:\text{prime}} \mathfrak{p}$. (another proof)

Pf: "$\subseteq$" clear

"$\supseteq$" : $\forall\, s \in \bigcap_{\mathfrak{p}} \mathfrak{p}$ if $s \notin \sqrt{0}$

$\Rightarrow 0 \notin \{1, s, s^2, \dots\} =: S$

$\Rightarrow S^{-1}A \neq 0$

$\Rightarrow \exists\, \mathfrak{q} \triangleleft S^{-1}A$: prim

$\Rightarrow \exists\, \mathfrak{p} \triangleleft A$ prime $\mathfrak{p} \cap S \neq \phi$

$\Rightarrow s \notin \mathfrak{p}$ ↯

<u>Cor</u> 3.12 $S^{-1}\,\mathrm{Nil}(A) = \mathrm{Nil}(S^{-1}A)$

<u>Cor</u> 3.13. $\{\mathfrak{p}' \triangleleft A \text{ prime} \mid \mathfrak{p}' \subset \mathfrak{p}\} \overset{1:1}{\longleftrightarrow} \{\mathfrak{q} \triangleleft A_{\mathfrak{p}} \text{ prime}\}$

Pf: $S := A \setminus \mathfrak{p}$

$A \rightsquigarrow A_{\mathfrak{p}}$ $\qquad \{\mathfrak{p}' \mid \mathfrak{p}' \subset \mathfrak{p}\}$

$A \rightsquigarrow A/\mathfrak{p}$ $\qquad \{\mathfrak{p}' \mid \mathfrak{p} \subset \mathfrak{p}'\}$

Fact : $\quad \mathrm{Frac}(A/\mathfrak{p}) \cong A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$

residue field at $\mathfrak{p}$

Prop 3.14 $\quad M =$ f.g. A-module. Then

$$S^{-1}(\mathrm{Ann}(M)) = \mathrm{Ann}(S^{-1}M)$$

Pf: induction on the number of generators.

1° $M \cong A/\mathfrak{a} \Rightarrow \begin{cases} \mathrm{Ann}(M) = \mathfrak{a} \\ S^{-1}M = S^{-1}A/S^{-1}\mathfrak{a} \Rightarrow \mathrm{Ann}(S^{-1}M) = S^{-1}\mathfrak{a} \end{cases}$

$$\Rightarrow \mathrm{Ann}(S^{-1}M) = S^{-1}\mathfrak{a} = S^{-1}\mathrm{Ann}(M)$$

2° $\quad S^{-1}(\mathrm{Ann}(M+N)) = S^{-1}(\mathrm{Ann}(M) \cap \mathrm{Ann}(N))$

$$= S^{-1}(\mathrm{Ann}(M)) \cap S^{-1}(\mathrm{Ann}(N))$$

$$= \mathrm{Ann}(S^{-1}M) \cap \mathrm{Ann}(S^{-1}N)$$

$$= \mathrm{Ann}(S^{-1}M \cap S^{-1}N)$$

$$= \mathrm{Ann}(S^{-1}(M+N)) \qquad \square$$

<u>Cor</u> 3.15 $N, P \subseteq M$ submodules. P = f.g. Then

$$S^{-1}(N:P) = S^{-1}N : S^{-1}P$$

Pf: $S^{-1}(N:P) \overset{2.2}{=} S^{-1}\left(\mathrm{Ann}\left((N+P)/N\right)\right)$

$$\overset{3.14}{=} \mathrm{Ann}\left(S^{-1}((N+P)/N)\right)$$

$$= \mathrm{Ann}\left((S^{-1}N + S^{-1}P)/S^{-1}N\right)$$

$$\overset{2.2}{=} (S^{-1}N : S^{-1}P) \qquad \square$$

<u>Prop 3.16</u> $A \xrightarrow{f} B$ ring hom. $\mathfrak{p} \triangleleft A$ prime. Then

$\mathfrak{p}$ = contraction of some <u>prime</u> ideal $\mathfrak{q} \triangleleft B$ $\iff$ $\mathfrak{p}^{ec} = \mathfrak{p}$.

$\Updownarrow$

$\mathfrak{p}$ = contraction of some ideal.

Pf: $\Rightarrow$) $\mathfrak{p} = \mathfrak{q}^c \Rightarrow \mathfrak{p} = \mathfrak{q}^c = \mathfrak{q}^{cec} = \mathfrak{p}^{ec}$ (1.17)

$\Leftarrow$) $\mathfrak{p} = (\mathfrak{p}^e)^c$ $\quad \mathfrak{q} := \mathfrak{p}^e$ $\times$ (not prime)

$$S := f(A - \mathfrak{p}) \subseteq B$$

$$\mathfrak{p}^e \cap S = \phi \quad \left(f(a) \in \mathfrak{p}^e \Rightarrow a \in \mathfrak{p}^{ec} = \mathfrak{p} \text{ } \lightning \right)$$

$$\mathfrak{p}^e \subset \mathfrak{m} \subset S^{-1}B$$

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & S^{-1}B \\ \mathfrak{p} & \longmapsto & \mathfrak{p}^e & \longmapsto & S^{-1}\mathfrak{p}^e \\ & & & & \cap \\ ? & \longleftarrow\!\shortmid & \mathfrak{q} & \longleftarrow\!\shortmid & \mathfrak{m} \\ \shortparallel & & & & \\ \mathfrak{p}\,! & & & & \end{array}$$

$$\mathfrak{q} \supseteq \mathfrak{p}^e \;\&\; \mathfrak{q} \cap S = \phi$$

$$\Rightarrow \mathfrak{q}^c = \mathfrak{p}$$